Topology qualifying examination, August 2020

The rules:

This is a closed book exam. Do not use any resources except your brain, pen or pencil, and scratch paper. If you prefer to typeset your work in LaTeX, close all other applications.

You need to complete any 4 of 5 problems. If you do all of them, indicate which problems should be graded. The exam is passed if these 4 problems are solved.

The standards of exposition are the same as for typeset mathematical texts. The notations and formulas should be explained, except the standard ones. Pictures are OK, but a picture without explanations does not prove anything.

Your claims should be justified by direct proofs or references to well-known basic theorems.

References such as “by Theorem 7.7 in Johnson’s book” are not acceptable. Do not use any advanced tools beyond elementary theories of fundamental groups and covering spaces, and elementary homology theory. Even if you studied Hatcher’s textbook, do not use Δ-complexes.

Problems

Problem 1. Let

$$W = \bigvee_{i=1}^{n} S^1_i$$

be the wedge of n copies S^1_i of the circle S^1, and let $X = W \times (0, 1)$. For every point $x \in X$ find the fundamental group of $X \smallsetminus \{x\}$.

Problem 2. Let F_3 be a free group with generators a, b, c. Use the theory of covering spaces to prove that $a^2 b^3 c^4 a \notin G$ for some subgroup G of finite index in F_3.

Problem 3. Let X be a topological space such that the universal covering space of X is homeomorphic to S^m for some $m > 1$. Let $\mathbb{T}^n = S^1 \times S^1 \times \ldots \times S^1$ be the product of n circles. Prove that every continuous map $X \longrightarrow \mathbb{T}^n$ is homotopic to a constant map.

Problem 4. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. Let us consider $\mathbb{R}^n$ as the subspace of $\mathbb{R}^{n+1}$ consisting of points with the last coordinate equal to 0. Then $S^{n-1} \subset S^n$ and S^{n-1} divides S^n into two hemispheres S^n_+ and S^n_-. Consider continuous maps

$$f\colon S^n \longrightarrow S^n$$

such that $f(S^n_+) \subset S^n_+$ and $f(S^n_-) \subset S^n_-$. Such a map f defines a map

$$f'\colon S^{n-1} \longrightarrow S^{n-1}\,.$$

(a) Use Mayer-Vietoris theorem to prove that for $n > 1$ the degrees of f and f' are equal.
(b) Prove that if $n > 1$, then for every integer d there exists such a map f of degree d.
(c) What degrees are possible for $n = 1$?

Problem 5. Let $\Delta^n \subset \mathbb{R}^{n+1}$ be the standard n-simplex, $\partial\Delta^n$ be its boundary, and let $\sigma = \mathrm{id}\colon \Delta^n \longrightarrow \Delta^n$ be the identity map considered as a singular simplex in Δ^n. Explain why $\partial\sigma$ can be considered as a singular chain in the boundary $\partial\Delta^n$ (the boundary $\partial\Delta^n$ is the union of all faces of Δ^n). Prove that this chain is a cycle and use Mayer-Vietoris theorem to prove that the homology class of this cycle is a non-zero element of the homology group $H_{n-1}(\partial\Delta^n)$. *Do not use simplicial or cellular homology!*